Statistical Extreme Value Theory (EVT) Part II

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NCAR

National Center for Atmospheric Research

Peaks Over Threshold (POT)

Poisson distribution applicable for modeling the *frequency* of exceeding a high threshold (i.e., low probability, rare, event).

What about the intensity of values that exceed the threshold?

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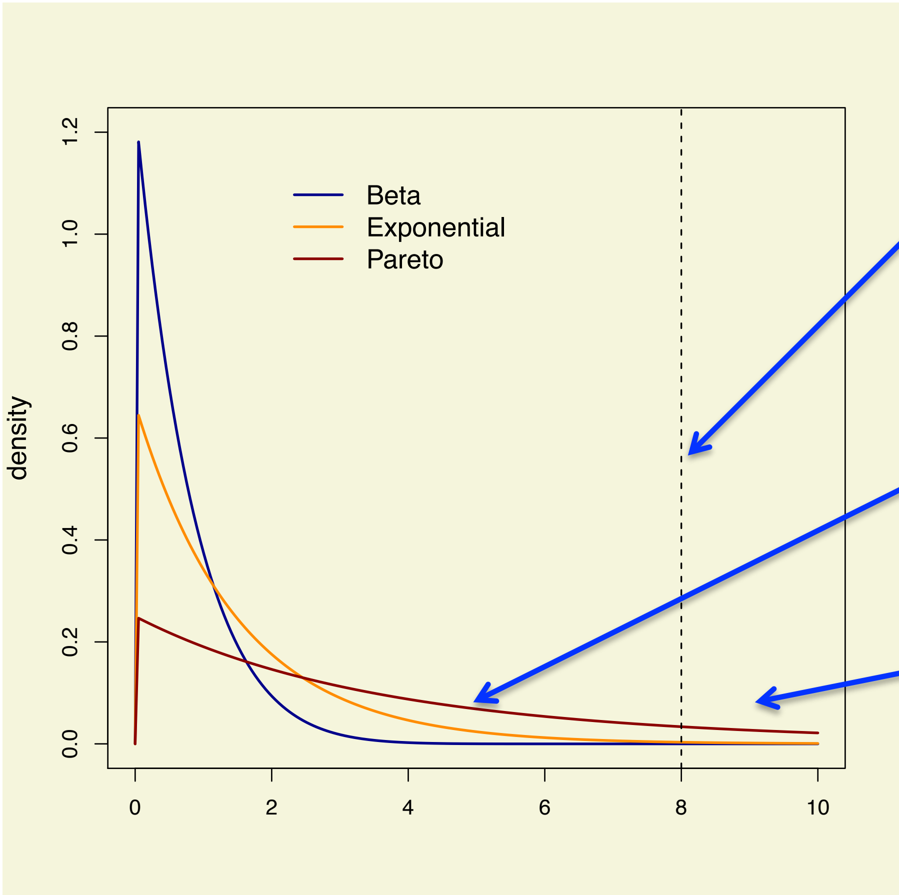
- X random variable
- Let Y = X – u, conditional on X > u, where u is a high threshold.
- Model the "excesses" over the threshold.
- Y has an approximate generalized Pareto (GP) distribution for high u with cdf:

$$H\left(y;\sigma(u),\xi\right)=1-\left[1+\xi\frac{y}{\sigma(u)}\right]^{-1/\xi},\ y>0,\ \xi\frac{y}{\sigma(u)}>0$$

$$\sigma(u)>0$$

Peaks Over Threshold (POT)

Three types of Extreme Value df's



Beta df; bounded upper tail

Exponential; light tail

Pareto; heavy tail

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- $\xi < 0$ yields the Beta cdf (bounded upper tail)
  upper bound at: $u - \sigma(u) / \xi$

- $\xi = 0$ yields the exponential cdf (“light” upper tail)

- $\xi > 0$ yields the Pareto cdf (“heavy” upper tail)

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Connection between GEV and GP families

- Maximum $M_n \leq u$ if no $X_i > u$, $i = 1, 2, \ldots, n$

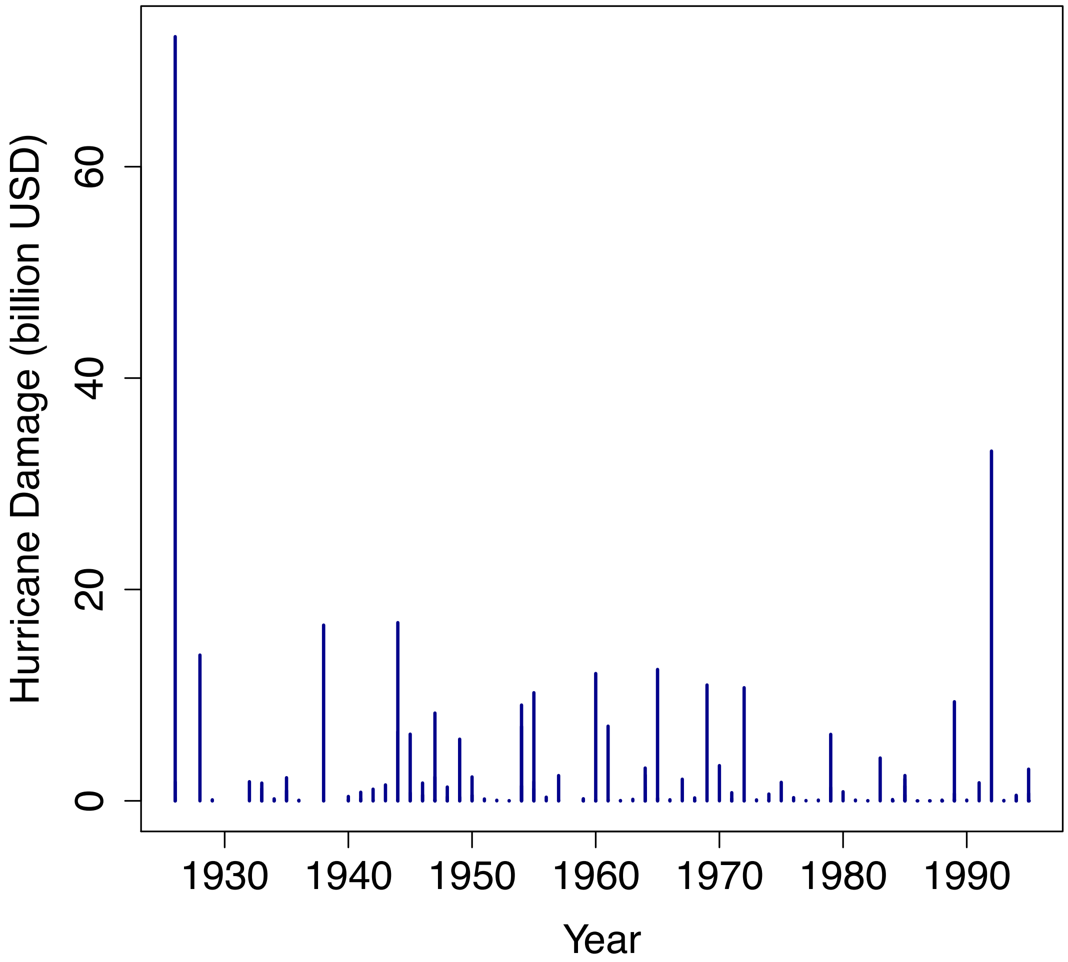
- "Memoryless" property of exponential distribution
  $\Pr\{ Y > y + y' \mid Y > y' \} = \Pr\{ Y > y \}$

- Stability of GP distribution
  - Lose memoryless property (need to rescale)

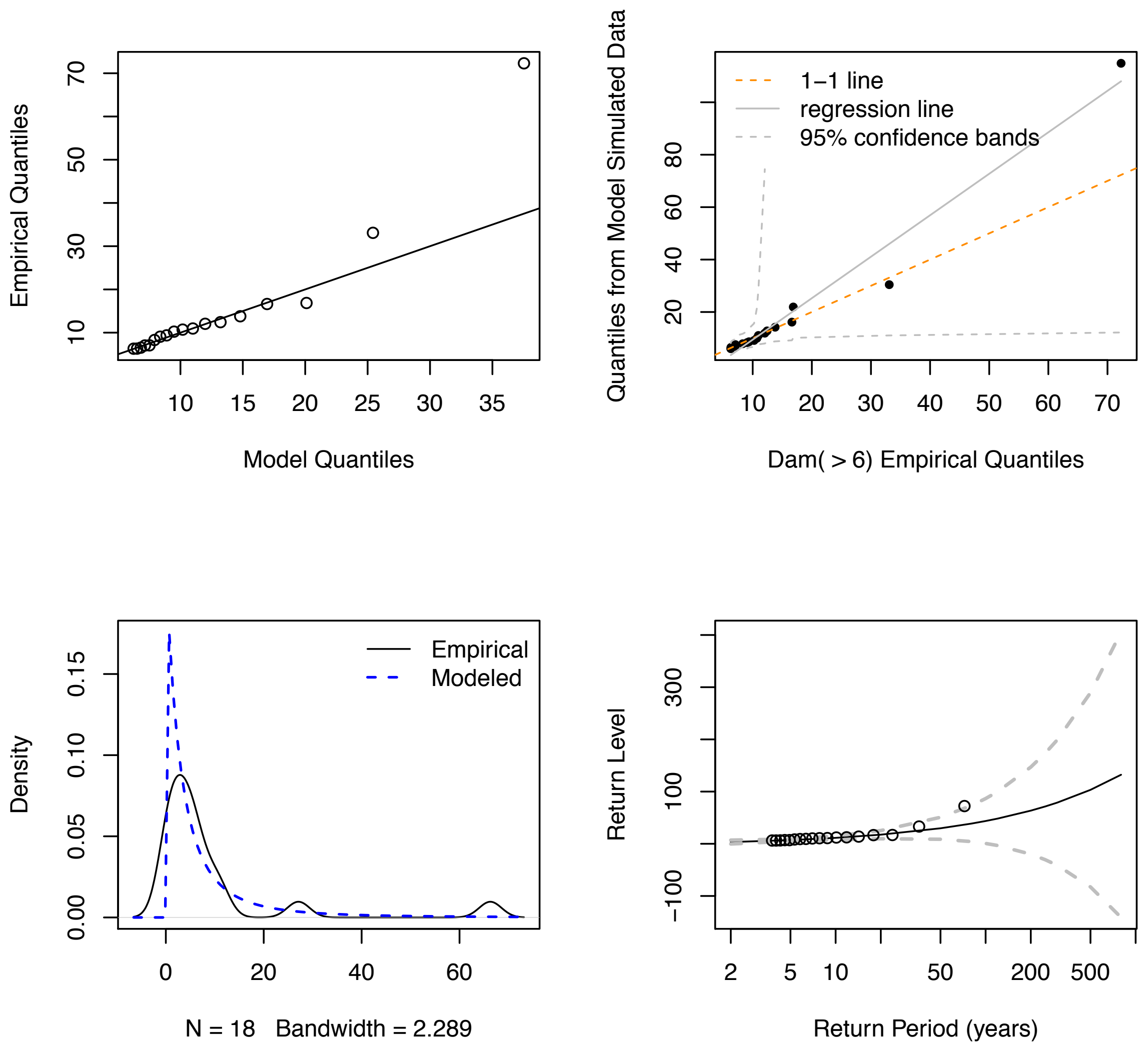
  - If $Y = X - u$ ($X > u$) has exact GP distribution with parameters $\sigma(u) > 0$ and ξ, then excess over a higher threshold $u' > u$ follows the GP distribution with parameters $\sigma(u') > 0$ and ξ, where

$$\sigma(u') = \sigma(u) + \xi(u' - u), \quad u' > u$$

Peaks Over Threshold (POT)



Peaks Over Threshold (POT)



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	95% lower CI	Estimate	95% upper CI
σ(u = 6 billion USD)	1.03 billion USD	4.59 billion USD	8.15 billion USD
ξ	-0.15	0.51	1.18
100-year return level (billion USD)	0.65 billion USD	43.64 billion USD	86.64 billion USD

Peaks Over Threshold (POT): GP return levels

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- First need quantile of GP cdf
  - $x_p = H^{-1}(1 - p; \sigma(u), \xi)$
  - $= (\sigma(u) / \xi) \times (p^{-\xi} - 1), 0 < p < 1$
- Complication: must account for the rate, ζ, of exceeding the threshold for interpretability, as well as the number of events per year, n_y.
  - replace p^{-1} with
  m = return period of interest $\times n_y \times \zeta$

Peaks Over Threshold (POT): GP return levels

Hurricane example

Lack of structure in data: some years have no hurricanes, some one, some two, etc.

Reasonable to use an average number per year in place of n_y.

Peaks Over Threshold (POT): Threshold Selection

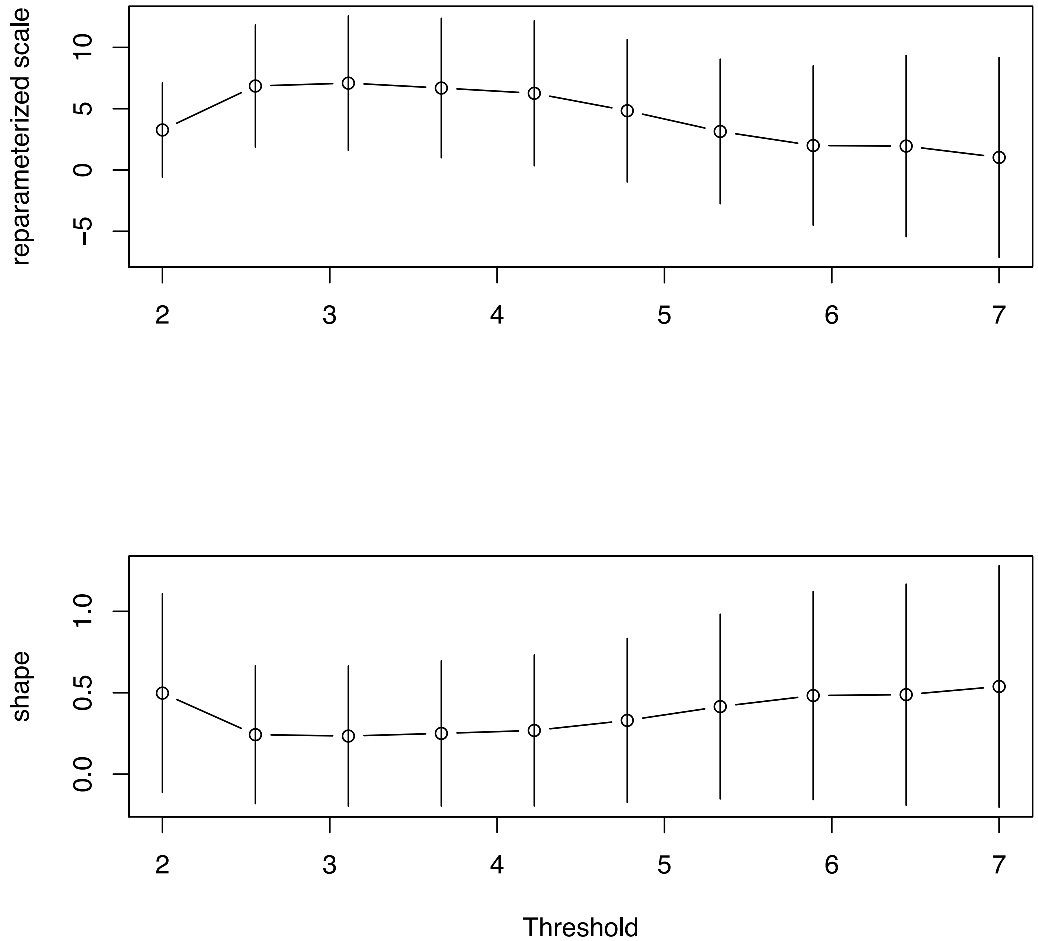
- The Bias-Variance Trade-Off
- Want a high threshold to obtain better GP approximation
- Want a lower threshold for more reliable estimation (more data!)
- Difficult to automate selection

Peaks Over Threshold (POT): Threshold Selection

Invariance of GP above threshold

- ξ does not change
- σ(u) is a function of the threshold
- The modification, $\sigma^* = \sigma(u) - \xi u$, is no longer a function of the threshold, and does not change
- Check for stability in parameter estimates as the threshold varies.

Peaks Over Threshold (POT): Threshold Selection



Peaks Over Threshold (POT): Dependence in Threshold Excess Data

- Remove dependence (e.g., decluster)
  - Runs declustering perhaps the simplest
- Model the dependence (e.g., through bivariate extreme value models)
- Concept of “extremal index”

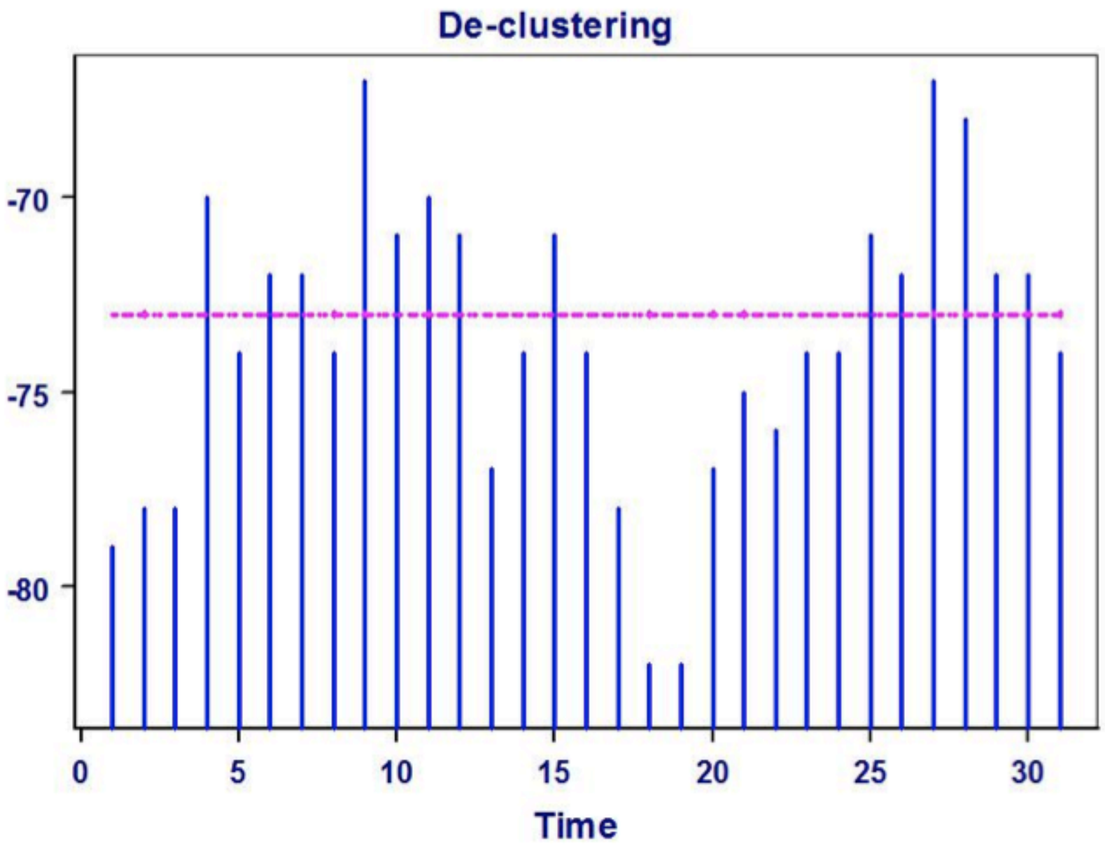
Peaks Over Threshold (POT): Dependence in Threshold Excess Data

Extremal Index, θ, $0 < \theta \leq 1$

Mean cluster size $\approx 1 / \theta$

$\theta = 1$ implies a lack of clustering at high levels, and degree of clustering at high levels increases as θ decreases

Peaks Over Threshold (POT): Dependence in Threshold Excess Data



Peaks Over Threshold (POT): Dependence in Threshold Excess Data

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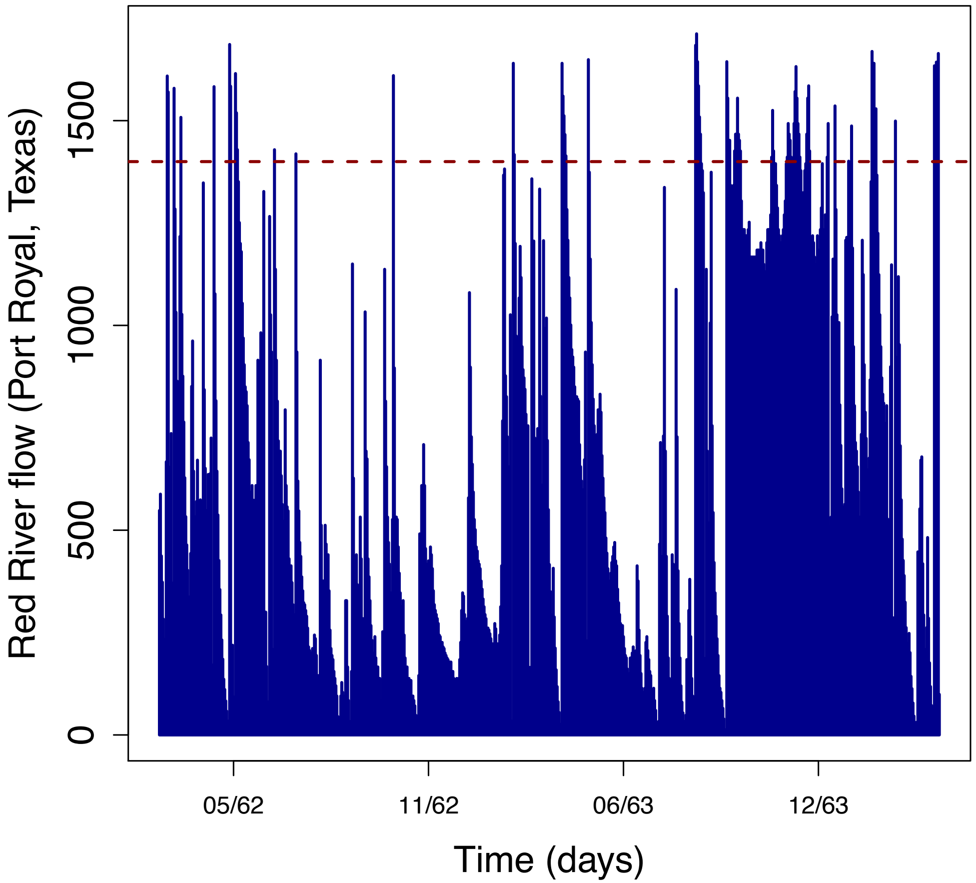
Full time range is 1 August 1961 to 28 September 2010

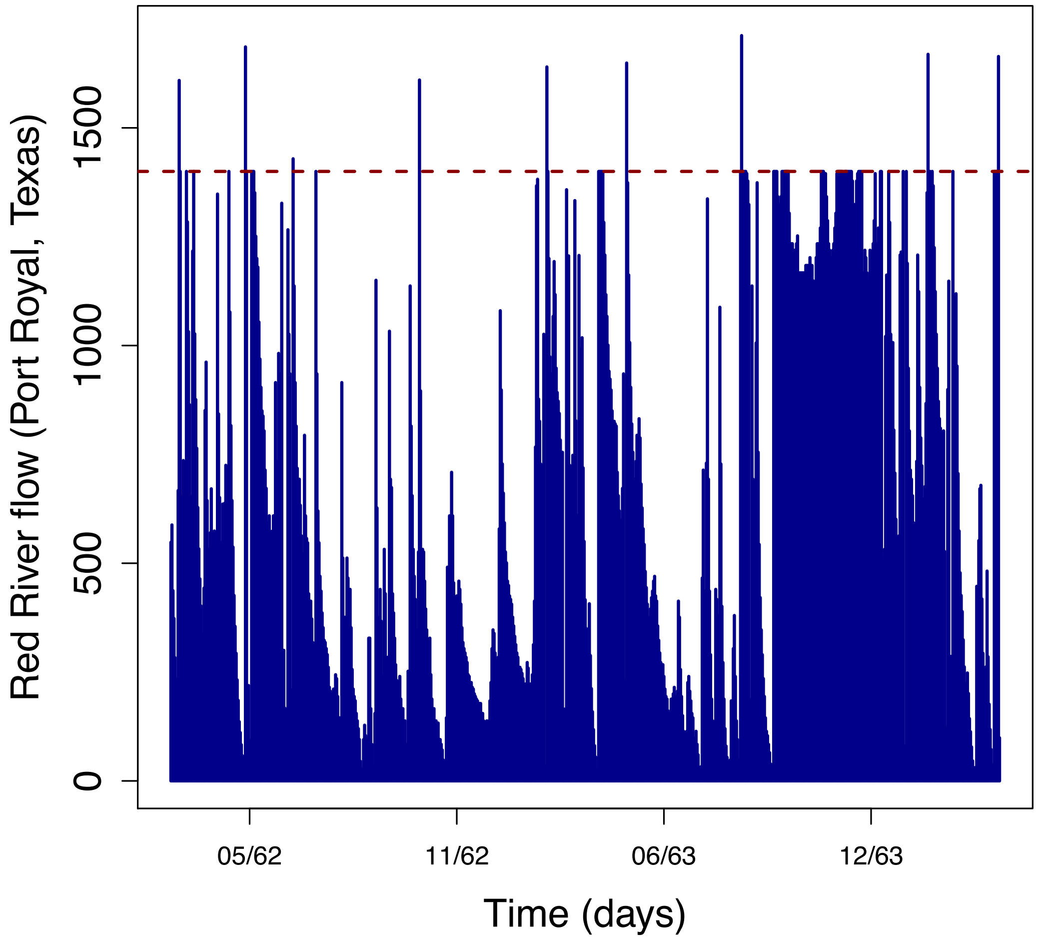


$\theta \approx 0.08$

r = 29

164 clusters

Peaks Over Threshold (POT): Dependence in Threshold Excess Data



θ ≈ 0.85

Peaks Over Threshold (POT): Modeling Frequency and Intensity

Poisson-GP Model

- Poisson process for exceeding a high threshold
  - Event: $X_t > u$
  - Rate parameter: λ
  - Number of events in [0, T] has Poisson distribution with parameeter λT
- GP distribution for excess over threshold
  - Excess $Y_t = X_t - u$ given $X_t > u$
  - Scale and Shape parameters

Peaks Over Threshold (POT): Modeling Frequency and Intensity

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Point Process (PP) representation

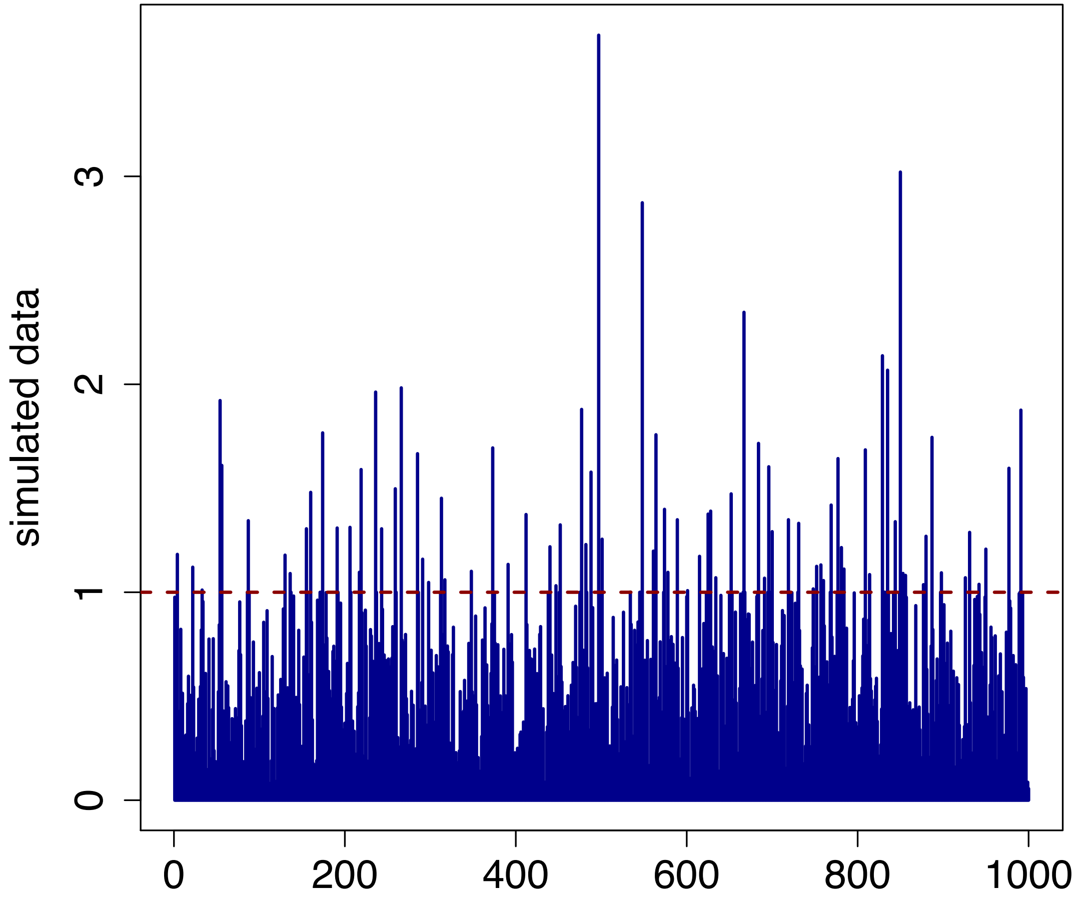
- Subsumes Poisson-GP model
- GEV parameterization (connection to GEV)
- Can relate parameters of GEV(μ, σ, ξ) to those of the PP(λ, $\sigma(u)$, ξ)
  - Shape parameter, ξ, is identical
  - $\ln \lambda = -(1/\xi) \ln(1 + \xi(u - \mu) / \sigma)$
  - $\sigma(u) = \sigma + \xi(u - \mu)$
- Need time scale, h, to take account of the block size for GEV (e.g., $h \approx 1/365.25$ for daily data and annual blocks)

Peaks Over Threshold (POT): Modeling Frequency and Intensity

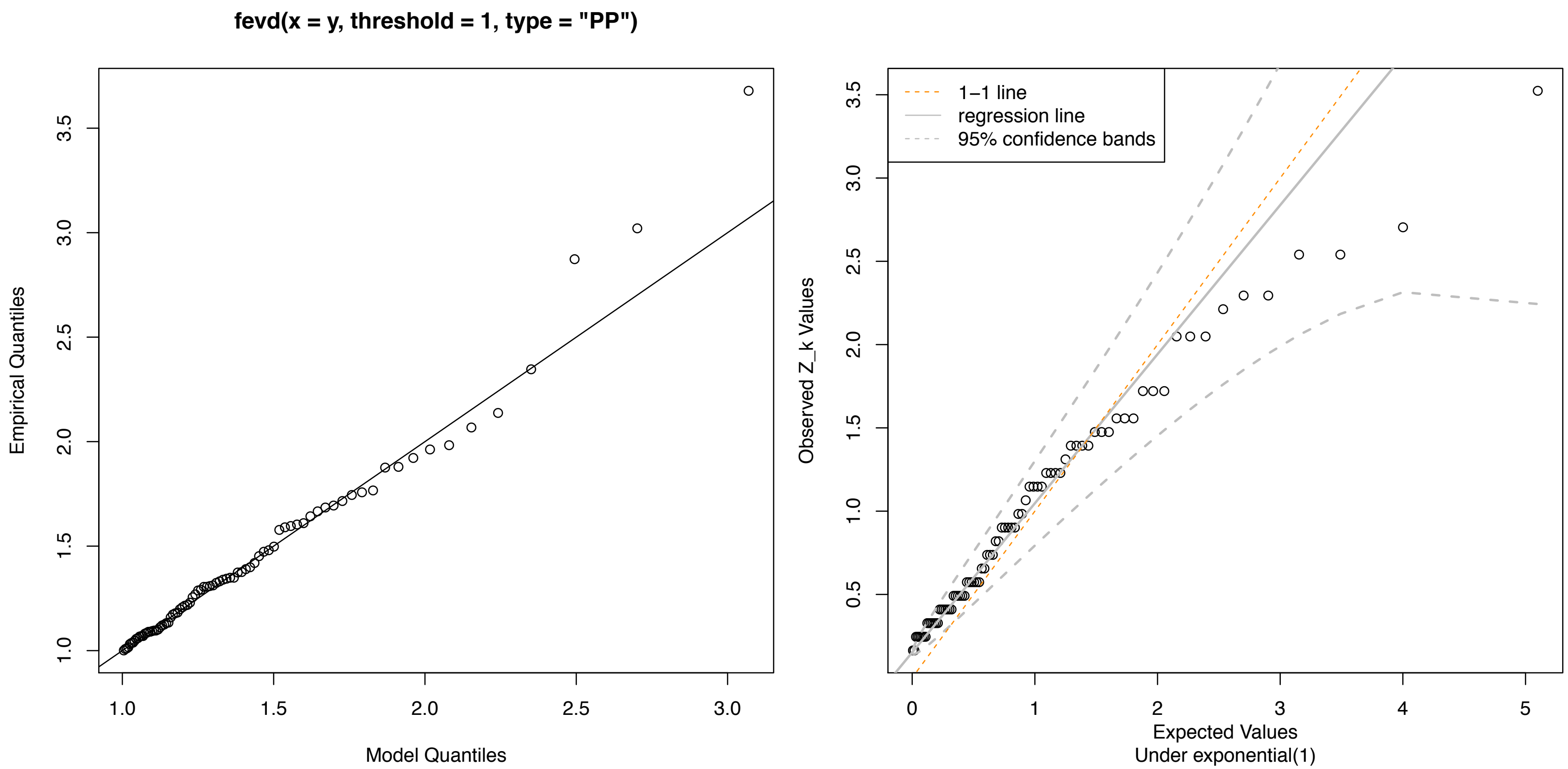
Point Process (PP) vs Poisson-GP

- Approaches are equivalent
- Poisson-GP model
  - Can obtain the same parameter estimates indirectly through the relationships between the two parameterizations
- Point Process
  - More convenient to quantify total uncertainty
  - Easier to interpret (eliminates dependence of parameters on the threshold)

Peaks Over Threshold (POT): Modeling Frequency and Intensity



Peaks Over Threshold (POT): Modeling Frequency and Intensity



Intensity Frequency

Peaks Over Threshold (POT): Modeling Frequency and Intensity

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Poisson-GP estimation method

	95% lower CI	Estimate	95% upper CI
λ	0.07	0.08	0.10
σ(u)	0.28	0.40	0.53
ξ	-0.16	0.07	0.29

PP estimation method

	95% lower CI	Estimate	95% upper CI
μ	2.06	2.53	3.01
σ	0.20	0.51	0.82
ξ	-0.16	0.07	0.30

Peaks Over Threshold (POT): Modeling Frequency and Intensity

Verify that the two sets of parameter estimates are consistent

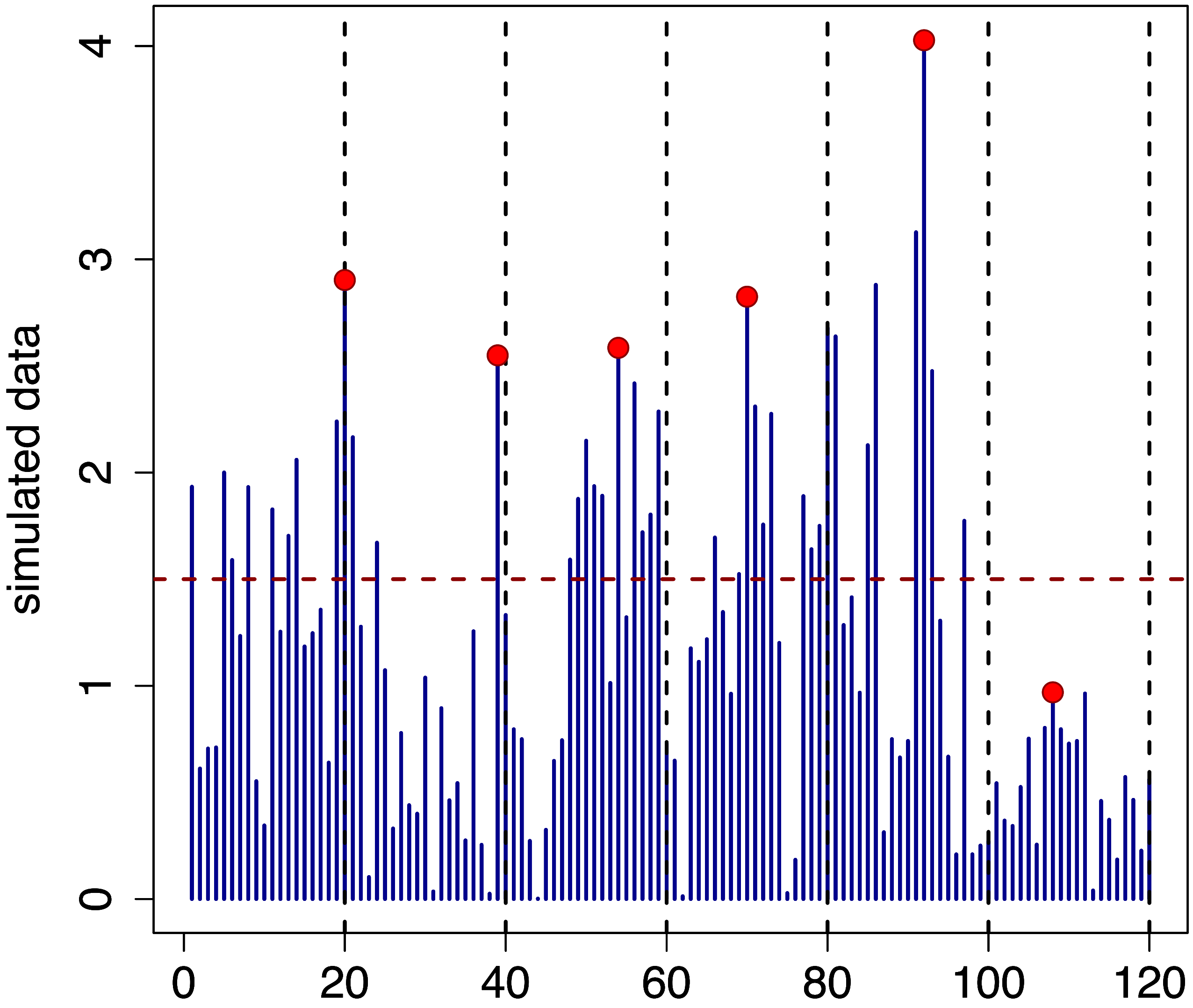
$\ln \lambda = -(1 / \xi) \ln(1 + \xi(u - \mu) / \sigma \approx 0.009$

$\lambda \approx 29.93$ (estimated on an annual maximum scale)

$0.08 * 365.25 \approx 29.22$

$\sigma(u) = \sigma + \xi(u - \mu) \approx 0.4$

POT vs Block Maxima



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Conclusion of Part II: Questions?