An improved approximation for the moist-air entropy potential temperature θ_s

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1 Motivations

The moist-air entropy is defined in Marquet (2011, hereafter M11) by $\boxed{s = s_{ref} + c_{pd}\,\ln(\theta_s)}$ in terms of two constant values (s_{ref}, c_{pd}) and a potential entropy temperature denoted by θ_s. It is shown in M11 that a quantity denoted by $(\theta_s)_1$ plays the role of a leading order approximation of θ_s.

The aim of this note is to demonstrate in a more rigorous way that $(\theta_s)_1$ is indeed the leading order approximation of θ_s, and to derive a second order approximation which may be used in computations of values, gradients or turbulent fluxes of moist-air entropy. Some impacts of this second order approximation are described in this brief version of a note to be submitted to the QJRMS.

2 Definition of θ_s and $(\theta_s)_1$

The potential temperature θ_s is defined in M11 by

$$\boxed{\theta_s = (\theta_s)_1 \left(\frac{T}{T_r}\right)^{\lambda\, q_t} \left(\frac{p}{p_r}\right)^{-\kappa\,\delta\, q_t} \left(\frac{r_r}{r_v}\right)^{\gamma\, q_t} \frac{(1+\eta\, r_v)^{\,\kappa\,(1+\delta\, q_t)}}{(1+\eta\, r_r)^{\,\kappa\,\delta\, q_t}}}$$

$$\text{where } \boxed{(\theta_s)_1 = \theta_l\ \exp(\Lambda_r\, q_t)\ .} \qquad (1)$$

This definition of θ_s is rather complex, but the more simple quantity $(\theta_s)_1$ was considered in M11 as a leading order approximation of θ_s, where the Betts' potential temperature is written as $\theta_l = \theta\ \exp[\,-\,(L_v\, q_l + L_s\, q_i)\,/\,(c_{pd}\, T)\,]$.

The total water specific content is $q_t = q_v + q_l + q_i$ and r_v is the water vapour mixing ratio. Other thermodynamic constants are: $R_d \approx 287$ J K^{-1}, $R_v \approx 461.5$ J K^{-1}, $c_{pd} \approx 1005$ J K^{-1}, $c_{pv} \approx 1846$ J K^{-1}, $\kappa = R_d/c_{pd} \approx 0.286$, $\lambda = c_{pv}/c_{pd} - 1 \approx 0.838$, $\delta = R_v/R_d - 1 \approx 0.608$, $\eta = R_v/R_d \approx 1.608$, , $\varepsilon = R_d/R_v \approx 0.622$ and $\gamma = R_v/c_{pd} \approx 0.46$.

The term

$$\boxed{\Lambda_r = [\,(s_v)_r - (s_d)_r\,]\,/\,c_{pd}\ \approx\ 5.87} \qquad (2)$$

depends on reference entropies of dry air and water vapour at $T_r = 0$ C, denoted by $(s_d)_r = s_d(T_r, e_r)$ and $(s_v)_r = s_v(T_r, p_r - e_r)$, where $e_r = 6.11$ hPa is the saturating pressure at T_r and $p_r = 1000$ hPa. The two reference entropies $(s_v)_r \approx 12673$ J K^{-1} and $(s_d)_r \approx 6777$ J K^{-1} are computed in M11 from the Third Law of thermodynamics, leading to $\Lambda_r \approx 5.87$. The reference mixing ratio is $r_r = \varepsilon\, e_r\,/\,(p_r - e_r) \approx 3.82$ g kg^{-1}.

3 Computations of Λ_s and $\Lambda_\star$

Let us define Λ_s by the formula $\theta_s = \theta_l\ \exp(\Lambda_s\, q_t)$ where θ_s, θ_l and q_t are known quantities, and thus by

$$\Lambda_s = \frac{1}{q_t}\ \ln\!\left(\frac{\theta_s}{\theta_l}\right). \qquad (3)$$

The aim is to compute Λ_s from (3) for a series of 16 vertical profiles of stratocumulus and cumulus with varying values of q_t, θ_s and θ_l, in order to analyse the discrepancy of Λ_s from the constant value $\Lambda_r \approx 5.87$ given by (2).

A trial and error process has shown that plotting Λ_s against $\ln(r_v)$ leads to relevant results (see Fig.1). Clearly, all stratocumulus and cumulus profiles are nearly aligned along a straight line with a slope of about -0.46. It is likely that this slope must correspond to $-\gamma$. This linear law appears to be valid for a large range of r_v (from 0.2 to 24 g kg^{-1}).

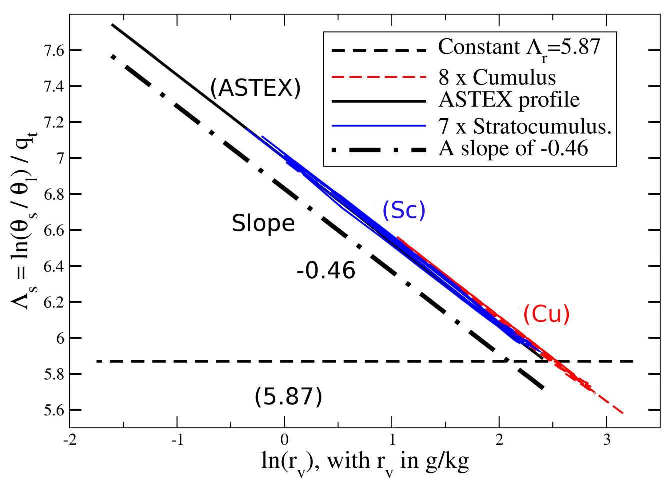

Figure 1: *A plot of Λ_s against $\ln(r_v)$ for 8 cumulus (dashed red), 7 stratocumulus (solid blue) and ASTEX (solid black) vertical profiles. The constant value $\Lambda_r \approx 5.87$ corresponds to the horizontal dashed black line. An arbitrary line with a slope of -0.46 is added in dashed-dotted thick black line*

It is then useful to find a mixing ratio $r_\star$ for which

$$\Lambda_s\ \approx\ \Lambda_\star\ =\ 5.87\ -\ 0.46\ \ln\left(r_v/r_\star\right) \qquad (4)$$

hold true, where $r_\star$ will play the role of positioning the dashed-dotted thick black line of slope $-\gamma \approx -0.46$ in between the cumulus and stratocumulus profiles. This corresponds to a linear fitting of r_v against $\exp[\,(\Lambda_r - \Lambda_s)/\gamma\,]$, $r_\star$ being the average slope of the scattered data points. It is shown in Fig.2 that the value $r_\star \approx 12.4$ g kg^{-1} corresponds to a relevant fitting of all cumulus and stratocumulus vertical profiles for a range of r_v up to 24 g kg^{-1}.

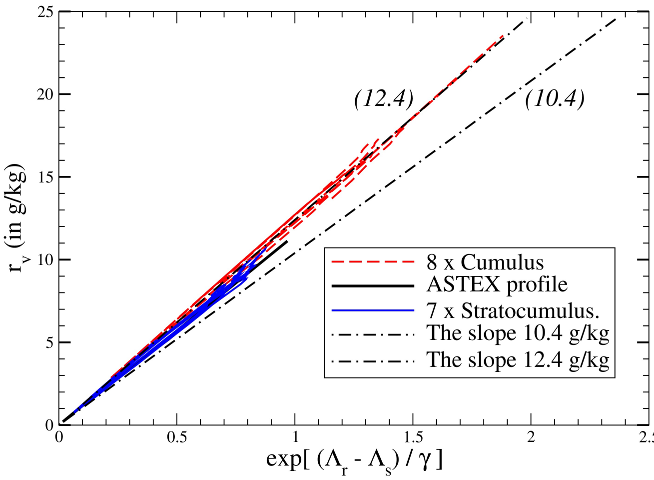

Figure 2: *Same as Fig.1, but with r_v plotted against the quantity* $\exp[\,(\Lambda_r - \Lambda_s)/\gamma\,]$. *Two lines of slope $r_* = 10.4$ and 12.4 g kg^{-1} are added as dashed-dotted thin black lines.*

It is shown in Fig.3 that Λ_s can indeed be approximated by $\Lambda_\star(r_v, r_*)$ given by (4), with a clear improved accuracy in comparison with the constant value $\Lambda_r \approx 5.87$ for a range of r_v between 0.2 and 24 g kg^{-1}. Curves of $\Lambda_\star(r_v, r_*)$ with $r_* = 10.4$ and 12.4 g kg^{-1} (solid black lines) both simulate with a good accuracy the non-linear variation of Λ_s with r_v, and both simulate the rapid increase of Λ_s for $r_v < 2$ g kg^{-1}.

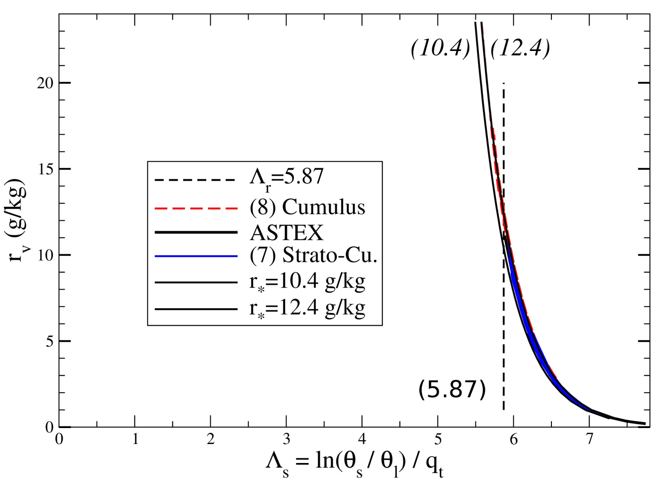

Figure 3: *Same as Fig.1, but with r_v plotted against the quantity Λ_s. The two thin black lines correspond to (4) with $r_* = 10.4$ or 12.4 g kg^{-1}.*

4 Mathematical computation of $\Lambda_\star$

It is important to confirm, by using mathematical arguments, that $(\theta_s)_1$ corresponds to the leading order approximation of θ_s, and that the slope of $-\gamma \approx -0.46$ (with $r_* \approx 10.4$ or 12.4 g kg^{-1}) corresponds to a relevant second order approximation for θ_s. These results are briefly mentioned in Marquet and Geleyn (2015).

First- and second-order approximations of θ_s can be derived by computing Taylor expansions for most of terms in the first formula recalled in Section 2.

The main result is that the term $(r_r/r_v)^{(\gamma\, q_t)}$ is exactly equal to $\exp[\,-(\gamma\, q_t)\,\ln(r_v/r_r)\,]$.

Then, the first order expansion of $(1+\eta r_v)^{[\,\kappa\,(1+\delta\, q_t)\,]}$ for small $r_v \approx q_t$ is equal to $\exp(\,\gamma\, q_t)$, since $\gamma = \kappa\,\eta$. The last term $(1+\eta\, r_r)^{(\kappa\,\delta\, q_t)}$ leads to the higher order term $\exp[\,\gamma\,\delta\, q_t\, r_r\,] \approx \exp[\,O(q_t^2)\,]$, since $r_r \ll 1$ and $q_t \ll 1$. Other terms depending on temperature and pressure are exactly equal to $\exp[\,\lambda\, q_t\, \ln(T/T_r)\,]$ and $\exp[\,-\kappa\,\delta\, q_t\, \ln(p/p_r)\,]$.

The Taylor expansion of θ_s can thus be written as

$$\boxed{\theta_s \approx \theta\, \exp\left(-\,\frac{L_{\text{vap}}\, q_l + L_{\text{sub}}\, q_i}{c_{pd}\, T}\right)\, \exp\left(\Lambda_*\, q_t\right)} \qquad (5)$$

$$\boxed{\times\, \exp\left\{ q_t \left[\lambda\, \ln\left(\frac{T}{T_r}\right) - \kappa\,\delta\, \ln\left(\frac{p}{p_r}\right)\right] + O(q_t^2) \right\}},$$

where $\boxed{\Lambda_* = \Lambda_r \;-\; \gamma\, \ln(r_v/r_*)}$

and $r_* = r_r \times \exp(1) \approx 10.4$ g kg^{-1} (see Figs.2 and 3).

The first order approximation of θ_s is thus given by the first line of (5) with $\Lambda_* = \Lambda_r$, namely by the expected $(\theta_s)_1$. An improved second order approximation is obtained by using Λ_* instead of Λ_r and by taking into account the small corrective term $-\,\gamma\, \ln(r_v/r_*)$.

Impacts of the second line of (5) with terms depending on temperature and pressure lead to higher order terms which must explain the fitted value $r_* \approx 12.4$ g kg^{-1} observed for usual atmospheric conditions.

5 Conclusions

It has been shown that it is possible to justify mathematically the first order approximation of θ_s derived in M11 and denoted by $(\theta_s)_1 \approx \theta_l\, \exp(\Lambda_r\, q_t)$, which depends on the two Betts' variables (θ_l, q_t) and $\Lambda_r \approx 5.87$, leading to $s \approx s_{ref} + c_{pd}\, \ln(\theta_l) + c_{pd}\, \Lambda_r\, q_t$.

A second order approximation is derived and compared to observed vertical profiles of cumulus and stratocumulus, leading to $s \approx s_{ref} + c_{pd}\, \ln(\theta_l) + c_{pd}\, \Lambda_*\, q_t$, where the second order term $\Lambda_* = \Lambda_r \;-\; \gamma\, \ln(r_v/r_*)$ depends on the constant $\Lambda_r \approx 5.87$, on the mixing ratio r_v, and on a tuning parameter $r_* \approx 12.4$ g kg^{-1}.

The use of the second order term Λ_* depending on the non-conservative variable $r_v \approx q_t - q_l - q_i$ can explain why it is needed to replace the Betts' potential temperature θ_l for computing flux of moist-air entropy: $\overline{w'\theta'_s} \approx \exp(\Lambda_*\, q_t)\, \overline{w'\theta'_l} + \Lambda_*\, \theta_s\, \overline{w'q'_t} - (\gamma\, q_t\, \theta_s/r_v)\, \overline{w'r'_v}$, the last term depending on $\overline{w'q'_t}$ minus $\overline{w'q'_l}$ or $\overline{w'q'_i}$.

References

• Marquet P. (2011). Definition of a moist entropic potential temperature. Application to FIRE-I data flights. *Q. J. R. Meteorol. Soc.* **137** (656) : p.768–791.

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