Representation of inhomogeneous, non-separable covariances by sparse wavelet-transformed matrices

Harald Anlauf[1], **Andreas Rhodin**[2] **and Werner Wergen**

Deutscher Wetterdienst, Offenbach, Germany

The purpose of this note is to advocate a new approach to the representation of an NMC- or ensemble-derived, inhomogeneous and non-separable forecast error covariance matrix $\mathbf{B}$ in a global data assimilation system. We use an explicit matrix representation of the covariances on the model grid or on an interpolated auxiliary grid. This approach is actually feasible because realistic NMC-derived covariance matrices have an extremely sparse representation in wavelet transformed space. In contrast to other methods – based on wavelet [1, 2, 3] or spectral transformations – our approach does not attempt to represent covariances by diagonal matrices in the transformed space, but allows for off-diagonal coefficients. The approach is based on orthogonal and bi-orthogonal wavelet transformations corresponding to wavelets with compact support, which can be implemented by fast algorithms [4].

The key idea of this approach is that the relevant information of covariance matrices is contained in a relatively small number of matrix elements. Coefficients accounting for irrelevant information – for instance correlations of small scale phenomena at large spatial separation – may be set to zero a priori. Thus the approach has filter characteristics which should also be useful when applied to forecast ensemble statistics, where filtering of noise – and where increasing the rank of the implied covariance matrices – is essential and generally performed by 'localization' procedures based on spatial separation only, but not on the scale of the phenomena.

To obtain a positive-definite, sparse representation of $\mathbf{B}$ we take the symmetric square root $\hat{\mathbf{L}}$ in the wavelet representation, set to zero all coefficients whose absolute value is smaller than a certain threshold, and recalculate $\hat{\mathbf{B}}$.

Figure 1(a) displays the entries of the wavelet-transformed matrix $\hat{\mathbf{L}}$ for the 500 hPa geopotential height correlation at 60°N obtained by the NMC method with zonal averaging and for 256 grid-points. Only a few coefficients of this matrix are substantially different from zero. Furthermore, these nonzero coefficients are essentially located in narrow bands along the diagonal – corresponding to correlations between wavelets of the same scale – and along off-diagonal branches corresponding to correlations between wavelets of different scales but at approximately the same location in grid space.

The grid space correlation function reconstructed from the truncated wavelet representation compares quite well with the original NMC correlation, see fig. 1(b). Of the order of 10 coefficients per grid-point are generally required for a good approximation of 1-dimensional correlations.

Our approach can be generalized to 2 and 3 dimensions. Here we expect of the order of 30 to 100 coefficients per grid-point to be sufficient for a good approximation of the covariance matrix (without taking advantage of redundancy due to symmetry and zonal

[1]Email: harald.anlauf@dwd.de

[2]Email: andreas.rhodin@dwd.de

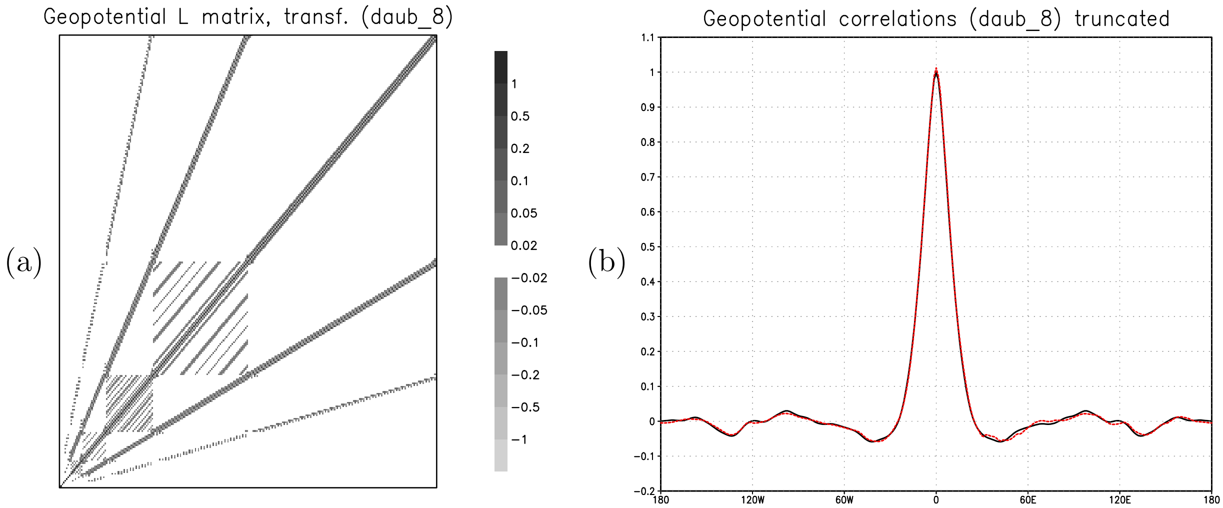

Figure 1: (a) Entries of the wavelet-transformed matrix $\hat{\mathbf{L}}$ for the 500 hPa NMC height correlation (256×256 elements, scaled for unit diagonal elements).
(b) Comparison of the NMC correlations (solid black line) with the approximation by the truncated wavelet expansion (keeping only coefficients with absolute value > 0.5% of the largest one, broken red line).

homogeneity). In 2 and 3 dimensions the suppression of sampling noise due to finite ensemble size and visible as random long-range correlations becomes very important. We are currently implementing and testing a 2d version in the 3D-Var-PSAS under development at DWD.

References

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