Preserving long-term accuracy in semi-implicit schemes for slightly compressible flow

Michael J.P. Cullen
Department of Mathematics, University of Reading, Reading U.K. RG6 6AX
m.j.p.cullen@reading.ac.uk

Semi-lagrangian methods for passive advection problems have the advantage that accuracy is maintained over a full timestep; since the equations are integrated along characteristics. The same applies to characteristic-based upwind Eulerian methods. In Riemann-solver based methods for compressible flows, accuracy is again maintained by transporting appropriate variables along the characteristics of the system.

In incompressible flows, this property will only be maintained if the trajectories describe a volume-preserving mapping from particle positions at departure points to their positions at arrival points, (Cullen 2002 section 3). No attempt is normally made to enforce this, since it requires implicit calculation of the departure points. The errors resulting from this can be calculated by diagnosing the flow Jacobian:

$$J = \det \frac{\partial(x_d, y_d, z_d)}{\partial(x_a, y_a, z_a)} \tag{1}$$

where suffices a and d refer to arrival and departure points respectively. Experiments in the model of Smolarkiewicz et al. (1999), reported by Cullen et al. (2001), showed that there could be substantial errors of up to 28% in a single time-step where the trajectories were deflected by orography.

In an incompressible problem, the errors can be reduced by iterative recalculation of the departure points. For a slightly compressible problem, the relevant case for the atmosphere, the equivalent procedure is to solve the continuity equation in Lagrangian form as

$$\rho_a = \rho_d J \tag{2}$$

where J is given by (1). This can be proved by arguments simlar to Cullen and Maroofi (2001). This equation has to be solved implicitly for numerical stability. We achieve this by using a predictor-corrector method, Cullen (2001). For example, for the calculation of the x-coordinate of the departure points, this gives

$$x_a = x_d^* + (u_d^t) + u_a^t)\delta t;\ x_a = x_d + (u_{d^*}^t + u_a^*)\delta t \tag{3}$$

In our implementation, J is approximated by finite differences. This is straightfoward in either two or three dimensions. In schemes designed to ensure formal conservation, e.g. Zerroukat et al. (2002), J has to be calculated by defining a departure-point control volume and integrating all variables over it. This is much harder in three dimensions than in two.

We demonstrate the viability of the method by using a slice version of the Met Office non-hydrostatic Unified Model (Cullen et al. 1997). The approximation to J is chosen so that the scheme is equivalent to the standard scheme in the limit of small timesteps. We solve a problem of flow at 10ms^{-1} over periodic hills of height 2000m and wavelength 5km. A 40×40 mesh is used. Since the scheme is equivalent to the standard scheme for short timesteps, we assess the relative performance by calculating the sensitivity to time step length. The table shows the r.m.s differences between various quantities between Courant numbers of 0.2 and 0.4; and between 0.4 and 0.8. Two versions of the model are compared:
A: Standard scheme with semi-Lagrangian advection of density and Ritchie method of calculating departure points.

B: Predictor-corrector scheme with solution of (2).
The table shows r.m.s differences calculated at two times. The first time is during the transient stage when the flow is being set up. The later time is when the steady state is being approached.

Table 1: Test of integration schemes.

Scheme/test	r.m.s. θ	rms u	rms w	r.m.s. θ	rms u	rms w
A .2/.4	.0180	.0142	.1324	.0343	.0212	.0408
B .2/.4	.0030	.0043	.0689	.0053	.0188	.0291
A .2/.8	.0283	.1535	.2477	.0855	.1099	.0951
B .2/.8	.0292	.0502	.2019	.0256	.0817	.1257

The results comparing the shorter timesteps show that the predictor-corrector scheme B gives much less timestep sensitivity than scheme A especially with shorter timesteps and integration periods. Plots of the results show that the results change substantially between Courant numbers of 0.4 and 0.8, so the differences may be less reliable than for the shorter timestep comparison. Experiemnts were also carried out with the predictor-corrector scheme using the standard form of the continuity equation and semi-lagrangian advection of density. This combination was found to be unstable, suggesting that (2) may be more robust than the standard formulation..

As used here, scheme B is twice as expensive as A. It is, however, likely that the cost of B could be substantially reduced by rationalising the iterations in the Helmholz solver and the departure point calculations. Cullen (2001) showed that the predictor-corrector scheme (without the use of the Lagrangian continuity equation (2)) could be cost-effective in the ECMWF model. These results suggest that the same may apply to the Met Office model.

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