Modeling of Covariance Matrices by Generalized Diffusion Operators – A Hierarchical Approach

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Because of its size, the forecast error covariance matrix $\mathbf{P}^f$ for the model prognostic variables $\mathbf{x}$ is generally modeled by choosing a different set of control variables $\mathbf{v}$, related to $\mathbf{x}$ by a linear transformation, $\mathbf{x} = \mathbf{L}\mathbf{v}$, so that the error covariance matrix for $\mathbf{v}$ is simple (e.g., diagonal). Generally the matrix $\mathbf{L}$ is split into a balance operator $\mathbf{K}_b$ and a remaining operator $\mathbf{P}_u^{f\,1/2}$.

$$\mathbf{L} = \mathbf{K}_b \mathbf{P}_u^{f\,1/2} \tag{1}$$

$\mathbf{K}_b$ accounts for the correlations between variables by transforming unbalanced variables $\mathbf{x}_u$ into their balanced counterparts (for instance by considering the geostrophic balance equation). $\mathbf{P}_u^f$ represents the univariate spatial correlations and thus is a block-diagonal matrix with block matrices $\mathbf{C}_\eta$. η denotes the unbalanced variables of the model (temperature, humidity, vorticity, divergence). The square root of the correlation matrices, $\mathbf{C}_\eta$, may be represented by appropriate operators or filter functions, $\mathbf{C}^{1/2}$:

$$\mathbf{C}_\eta = \mathbf{C}_\eta^{1/2} \mathbf{C}_\eta^{T/2} \tag{2}$$

A common choice for $\mathbf{C}$ is a spectral transform [1], implying horizontally homogeneous and isotropic error covariance functions. This choice has serious limitations, because error covariances are flow dependent, and therefore their representation in data assimilation systems should be spatially varying. An alternative for representing $\mathbf{C}$ is to use digital filters in grid point space [2]. Digital filters can be implemented efficiently for Gaussian covariance functions on regular orthogonal grids by an alternating direction approach. An efficient application of both methods on an icosahedral grid, as used by the global model GME [3] of DWD, is difficult.

Weaver and Courtier [4] have proposed modeling the filter functions $\mathbf{C}_\eta^{1/2}$ by application of a diffusion operator, i.e. integrating the diffusion equation over a time span T starting with the unbalanced fields $\eta(\tau = 0)$, with

$$\frac{\partial \eta}{\partial \tau} - \bigtriangledown \kappa \bigtriangledown \eta = 0 \;, \tag{3}$$

This operator has been generalized in order to model non Gaussian correlation functions. Here a different generalization is proposed:

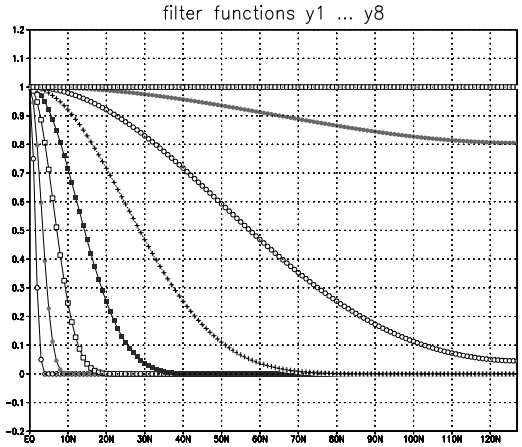
$$\frac{\partial \eta}{\partial t} - \bigtriangledown \kappa(\tau) \bigtriangledown \eta - c(\tau)\eta(0) = 0 \tag{4}$$

This formulation differs from (3) by the additional source term $c(\tau)\,\eta(0)$. For small values of τ, the coefficient $c(\tau)$ controls the shape of the correlation function on the large scales (long integration times $T{-}\tau$ of the diffusion equation), whereas for large values of τ (small $T{-}\tau$), it controls the shape on the fine scales. By choosing negative values of $c(\tau)$ for $\tau << T$ and positive values for $t \approx T$, correlation functions that are negative on the large scales can be modeled. By choosing anisotropic diffusion coefficients κ, anisotropic correlation functions are obtained.

The number of steps required to integrate equations (3) or (4) numerically is limited by stability criteria. However, the diffusion equation can be integrated efficiently using a multi-grid approach, by starting the integrating of equation (4) on a coarse grid for $\tau << T$ and then moving to finer grids for $\tau \approx T$. In praxis the algorithm is as follows:

1. Average the field $\eta(0)$ to the coarser grids.
2. Integrate the diffusion equation on the coarsest grid for a few (large) steps.
3. Interpolate the intermediate result to the next finer grid.
4. Integrate the diffusion equation for a few (smaller) steps. In practice the coefficients $c(\tau)$ and $\kappa(\tau)$ may be kept constant on each grid i, denoted c_i and κ_i.
5. Repeat steps 3 and 4 until the finest grid is reached.

Correlation functions of various shapes may be approximated by a linear combination of the filter functions y_i with appropriate coefficients c_i. The approach was tested on a 2-dimensional regular grid and is illustrated below. In this example 8 grids were used (with grid spacing differing by a factor of 2) until the coarsest grid was reached (one grid point only).



a) Filter functions y_i obtained by setting $c \neq 0$ on grid level i only.

b) Gaussian correlation function approximated by a linear combination of the functions $y_1 \ldots y_8$.
open circles: specified covariance function $\exp(-r^2/l^2)$
crosses: correlation function obtained by applying the approximated filter function.
filled circles: required filter function (square root of the above covariance function).
open squares: approximated filter function.
filled squares: difference of the exact and approximated filter function.

c) As b), but for a covariance function
$c = (\cos(rl_c) + sin(rl_c)/(ll_c))\exp(-r/l)$.

Up to now the multi-grid approach was motivated by the goal to yield an efficient operator representation of the filter functions. The approach may be modified by defining not only the coefficients c_i on the hierarchical grid, but the control variables $\mathbf{x}_i$ as well. Then, the control variables are representing analysis increments on different scales at different locations, and the transformation $\mathbf{L}$ behaves like a wavelet transformation.

References

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