Advanced ice sheet modeling: scalable parallel adaptive full Stokes solver and inversion for basal slipperiness and rheological parameters

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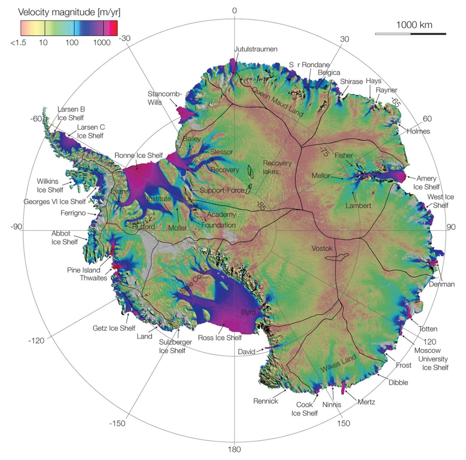
Summary

- Modeling the flow of polar ice sheets using the nonlinear 3D full Stokes equations requires scalable and efficient solvers as well as the capability to infer uncertain model parameters (i.e., basal boundary conditions and rheology parameters) from available observations.



Observed surface flow velocity rate in the Antarctic ice sheet (Rignot et. al, "Ice Flow of the Antarctic Ice Sheet", Science Express, Aug 2011.)

- We present a parallel, adaptive mesh, high-order finite element solver for the 3D full Stokes equations with Glen's flow law rheology.
- We also formulate and solve inverse problems to infer the basal slipperiness and rheological parameters from surface observations.

Parallel adaptive full Stokes solver

Finite element discretization

- Velocity degrees of freedom: continuous approximation using tensor product polynomials Q^p (order 1, 2, and 3 used in results below), Legendre-Gauss-Lobatto (LGL) nodal basis.
- Pressure degrees of freedom: stabilized ($p = 1$), discontinuous approximation using polynomials P^{p-1}, or tensor product polynomials Q^{p-2}.

Parallel adaptive mesh refinement

- Ice sheet dynamics are characterized by localized flow features. We use adaptive mesh refinement (AMR) to resolve important features while avoiding a fine grid everywhere.
- The parallel AMR library `p4est` manages the data structures used and maintains load balance between processors.

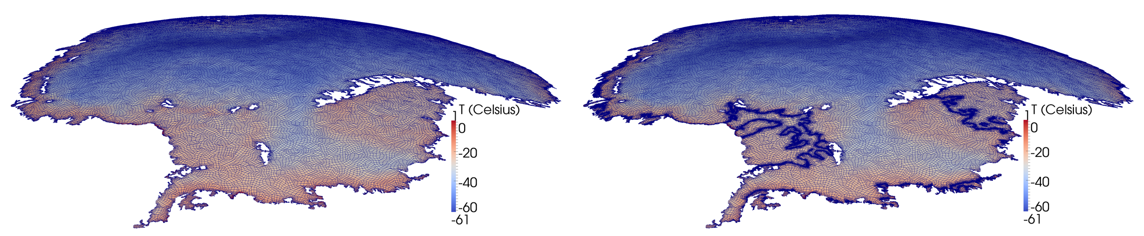
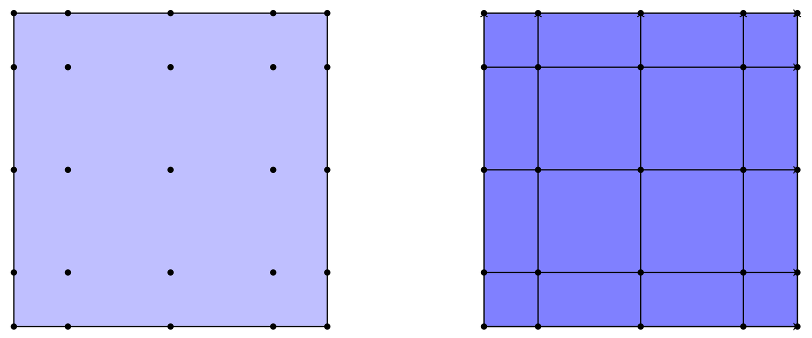


Figure: *Resolution of the grounding line with arbitrary precision. If the grounding line moves, the coarse mesh does not have to change.*

Scalable solver for the linear Stokes problem

We use Krylov subspace methods for symmetric indefinite systems of equations, either MINRES (low storage) or GMRES (admits nonsymmetric preconditioners). Convergence depends on ability to precondition two submatrices:

- The stress block $V(\eta, \beta)$, which operates on velocity and applies viscous stress and basal friction, is elliptic and sparse.
  - We use the algebraic multigrid (AMG) solver ML in Trilinos.
  - AMG has good parallel scalability and its quality as a preconditioner does not degrade for larger problems.
  - AMG is not as effective for matrices from higher-order elements, so for $p > 1$ we use the matrix from a mesh of trilinear elements using the same degrees of freedom as the higher-order mesh.



- The Schur complement of the full Stokes operator

$$S = -\nabla \cdot V^{-1} \nabla,$$

which operates on the pressure, is dense and is never formed. To precondition S, we use the least square commutator approximation

$$S^{-1} \approx -\Delta^{-1} \nabla \cdot V \nabla \Delta^{-1},$$

where the Laplacian on the pressure space is approximately inverted using AMG. Changing properties of V due to varying viscosity or basal friction are taken into account by S^{-1}.

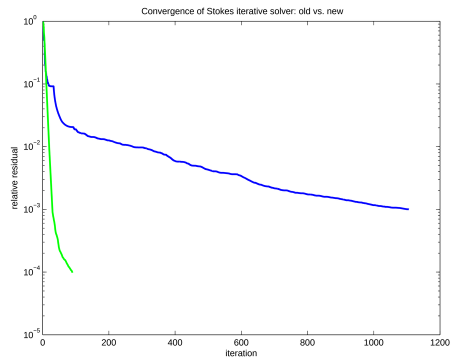


Figure: *Comparison of the convergence of two different numerical approaches to the same linear Stokes problem to demonstrate the advantage of advanced numerical methods. The problem is solved on the full Antarctic ice sheet with roughly 10 million velocity degrees of freedom, realistic variations in viscosity due to temperature and with large basal friction. The first method (blue) uses MINRES and a diagonal preconditioner for the Schur complement; the second method (green) uses GMRES with a block upper-triangular preconditioner and the least squares commutator to precondition the Schur complement.*

Inverse problems for a nonlinear Stokes ice sheet model

Ice sheet flow models contain parameters that are unknown and cannot be directly observed. Available observations can be used to infer these parameters by minimizing the following regularized misfit functional

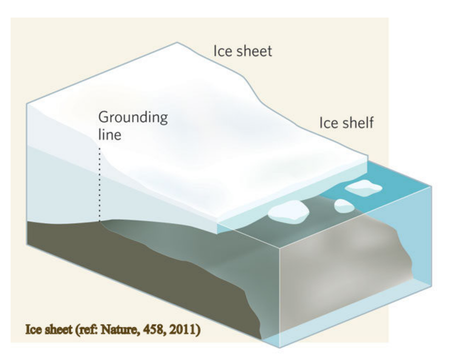
$$\min_{\beta, n} \mathcal{J}(\beta, n) := \underbrace{\frac{1}{2} \int_{\Gamma_t} \mathcal{B}(\boldsymbol{x})(\boldsymbol{u} - \boldsymbol{u}^{\text{obs}})^2 \, d\boldsymbol{s}}_{\text{misfit}} + \underbrace{\mathcal{R}(\beta)}_{\text{regularization}},$$

where the dependence of $\boldsymbol{u}$ on β is given by solving the Stokes system:

Forward nonlinear Stokes problem:

$$\begin{aligned} -\boldsymbol{\nabla} \cdot [2\eta(\boldsymbol{u}, n)\dot{\boldsymbol{\varepsilon}}_{\boldsymbol{u}} - \boldsymbol{I}p] &= \rho \boldsymbol{g} && \text{in } \Omega \\ \boldsymbol{\nabla} \cdot \boldsymbol{u} &= 0 && \text{in } \Omega \\ \boldsymbol{\sigma}_{\boldsymbol{u}} \boldsymbol{n} &= \boldsymbol{0} && \text{on } \Gamma_t \\ \boldsymbol{u} \cdot \boldsymbol{n} &= 0, && \text{on } \Gamma_b \\ \boldsymbol{T}\boldsymbol{\sigma}_{\boldsymbol{u}}\boldsymbol{n} + \beta |\boldsymbol{T}\boldsymbol{u}|^{m-1} \boldsymbol{T}\boldsymbol{u} &= \boldsymbol{0} && \text{on } \Gamma_b \end{aligned}$$

+ additional B.C.s



- $\boldsymbol{u}^{\text{obs}}$ observed surface flow velocity
- $\mathcal{B}(\boldsymbol{x})$ is an observation operator
- $\boldsymbol{u}$ ice flow velocity, p pressure
- $\eta(\boldsymbol{u}, n) = \frac{1}{2} A^{-\frac{1}{n}} \dot{\boldsymbol{\varepsilon}}_{\text{II}}^{\frac{1-n}{2n}}$ effective viscosity
- $\dot{\boldsymbol{\varepsilon}}_{\boldsymbol{u}} = \frac{1}{2}(\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}\boldsymbol{u}^T)$ strain rate tensor
- $\boldsymbol{\sigma}_{\boldsymbol{u}} = -\boldsymbol{I}p + 2\eta(\boldsymbol{u}, n)\dot{\boldsymbol{\varepsilon}}_{\boldsymbol{u}}$ stress tensor
- $\dot{\boldsymbol{\varepsilon}}_{\text{II}} = \frac{1}{2}\text{tr}(\dot{\boldsymbol{\varepsilon}}_{\boldsymbol{u}}^2)$ second invariant of the strain rate tensor
- ρ density, $\boldsymbol{g}$ gravity
- $\boldsymbol{n}$ unit normal vector
- β basal sliding coefficient
- m basal sliding exponent
- n flow-law exponent
- $\boldsymbol{T} = \boldsymbol{I} - \boldsymbol{n} \otimes \boldsymbol{n}$ tangential operator
- Γ_t and Γ_b top and base boundaries

Adjoint-based inexact Newton method for solution of the inverse problem

Given the current estimate of the inversion parameters (β, n), Newton's method iterates on the solution of a sequence of linear systems of the form

$$\mathcal{H}(\beta, n)(\tilde{\beta}, \tilde{n}) = -\mathcal{G}(\beta, n),$$

for the Newton direction $(\tilde{\beta}, \tilde{n})$, where

- $\mathcal{G}$ is the gradient of the regularized data misfit functional $\mathcal{J}$:

$$\mathcal{G}(\beta, n) := \begin{cases} -\nabla \cdot (\gamma_\beta \boldsymbol{T} \nabla \beta) + |\boldsymbol{T}\boldsymbol{u}|^{m-1} \boldsymbol{T}\boldsymbol{u} \cdot \boldsymbol{T}\boldsymbol{v} & \text{on } \Gamma_{\text{b}}, \\ (\gamma_\beta \boldsymbol{T} \nabla \beta) \cdot \overline{\boldsymbol{n}} & \text{on } \partial\Gamma_{\text{b}}, \\ -\nabla \cdot (\gamma_n \nabla n) + 2\frac{d\eta}{dn}(\boldsymbol{u}, n)\, \dot{\boldsymbol{\varepsilon}}_{\boldsymbol{u}} : \dot{\boldsymbol{\varepsilon}}_{\boldsymbol{v}} & \text{in } \Omega, \\ (\gamma_n \nabla n) \cdot \boldsymbol{n} & \text{on } \partial\Omega \end{cases}$$

  - β and n are model parameter fields
  - $\boldsymbol{u}$ (velocity) satisfies *nonlinear Stokes* equation
  - $\boldsymbol{v}$ (adjoint velocity) satisfies *adjoint Stokes* equation

Adjoint Stokes problem:

$$\begin{aligned} -\boldsymbol{\nabla} \cdot [\eta(\boldsymbol{u}, n)\mathsf{T}(\boldsymbol{\nabla}\boldsymbol{v} + \boldsymbol{\nabla}\boldsymbol{v}^T) - \boldsymbol{I}q] &= \boldsymbol{0} && \text{in } \Omega \\ \boldsymbol{\nabla} \cdot \boldsymbol{v} &= 0 && \text{in } \Omega \\ \boldsymbol{\sigma}_{\boldsymbol{v}} \boldsymbol{n} &= \boldsymbol{u}^{\text{obs}} - \boldsymbol{u} && \text{on } \Gamma_t \\ \boldsymbol{v} \cdot \boldsymbol{n} &= 0, && \text{on } \Gamma_b \\ \boldsymbol{T}\boldsymbol{\sigma}_{\boldsymbol{v}}\boldsymbol{n} + \beta|\boldsymbol{T}\boldsymbol{u}|^{m-1}\boldsymbol{T}\boldsymbol{v} + {} & && \text{on } \Gamma_b \\ \beta(m-1)|\boldsymbol{T}\boldsymbol{u}|^{m-3}(\boldsymbol{T}\boldsymbol{u} \otimes \boldsymbol{T}\boldsymbol{u})\boldsymbol{T}\boldsymbol{v} &= \boldsymbol{0} \end{aligned}$$

+ additional B.C.s, $\mathsf{T} = \mathsf{I} + \frac{1-n}{n} \frac{\dot{\boldsymbol{\varepsilon}}_{\boldsymbol{u}} \otimes \dot{\boldsymbol{\varepsilon}}_{\boldsymbol{u}}}{\dot{\boldsymbol{\varepsilon}}_{\boldsymbol{u}} : \dot{\boldsymbol{\varepsilon}}_{\boldsymbol{u}}}$

- $\mathcal{H}$ is the Hessian operator; the application of the Hessian to a vector requires solving two linear Stokes-like problems.

Performance of inexact Newton vs. nonlinear conjugate gradient methods

Comparison of iterations and number of Stokes solves required for convergence for different meshes:

			NCG (FR/PR)*		inexact Newton	
mesh	#dof	#par	#iter	# Stokes	#iter	# Stokes
10×10×2	6978	121	54/57	622/676	10 (41)	104
20×20×2	26538	441	57/58	620/572	10 (43)	108
40×40×2	103458	1681	99/87	674/648	10 (48)	118
80×80×2	408498	6561	125/102	664/623	10 (46)	114

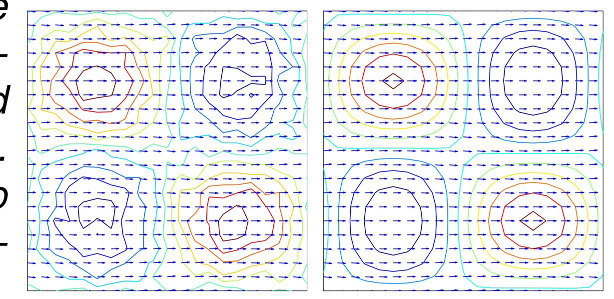
* Nonlinear Conjugate Gradient (FR (Fletcher-Reeves) / PR (Polak-Ribière))

** Iteration is terminated when the norm of the gradient drops below a relative tolerance of 10^{-5}.

The number of Newton inversion iterations and required Stokes solves are insensitive to the number of inversion parameters.

Inversion for the basal sliding coefficient β

Figure: *Observations of the surface velocity with signal-to-noise ratio of 100 (left) and reconstructed velocity (right). Contour lines correspond to the horizontal velocity component.*



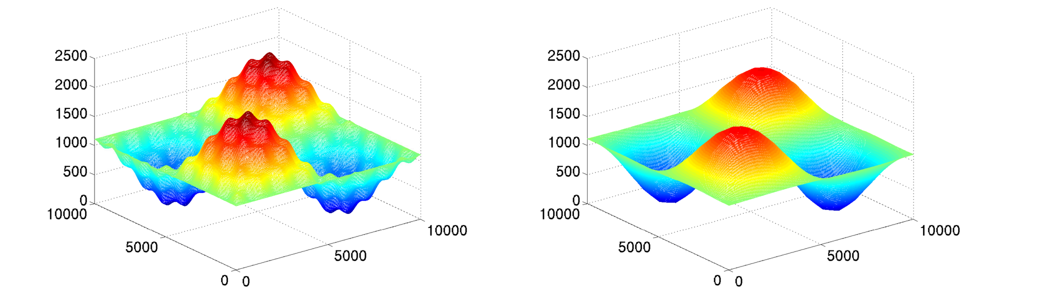
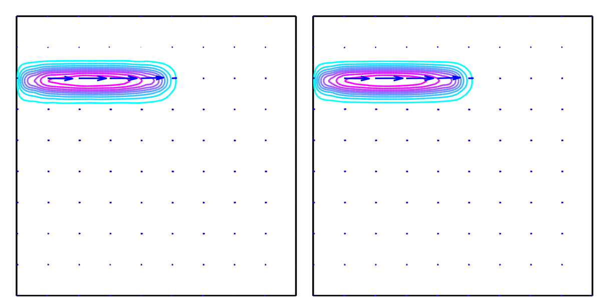


Figure: *Inversion for basal sliding coefficient β. Left: truth β field. Right: inverted β based on noisy observations. The small wavelength basal variations cannot be reconstructed from the noisy observations.*

Inversion for the rheology parameter n

Figure: *Noisy observations of the surface velocity (left) and reconstructed velocity (right). Contour lines correspond to the horizontal velocity component.*



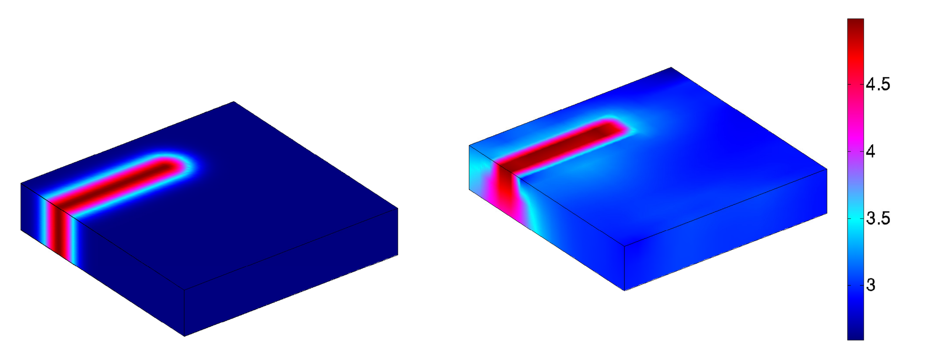


Figure: *Inversion for Glen's flow law exponent parameter n. Left: truth n field. Right: inverted n based on noisy observations.*

Conclusions

- Advanced numerical methods and adaptive mesh refinement make possible the efficient solution of continental-scale full Stokes ice sheet models with sufficient resolution.
- Smooth variations of basal slipperiness can be reconstructed from surface velocity observations.
- Volume rheology parameter reconstruction is a highly ill-posed problem that requires regularization/prior knowledge.
- Newton-type methods are crucial for large scale problems.

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