The mixed-phase version of moist-air entropy

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1 Motivations

The specific (per unit mass of *moist-air* entropy is defined in Marquet (2011, M11) by $s = s_{ref} + c_{pd} \ln(\theta_s)$, where s_{ref} and c_{pd} are two constants. The first- and second-order approximations $(\theta_s)_1$ and $(\theta_s)_2$ of the moist-air entropy potential temperature θ_s have been more recently derived in Marquet (2015, M15).

The aim of this note is to derive the *mixed-phase* version of θ_s, $(\theta_s)_1$ and $(\theta_s)_2$, namely if liquid water and ice are allowed to coexist, with possible under- or super-saturations, with possible supercooled water and with possible different temperatures for dry air and water vapour, on the one hand, condensed water and ice, on the other hand.

2 The mixed-phase definition of θ_s

The specific (per unit mass of *moist-air*) entropy given by (B.1) in M11 is equal to the sum

$$s \;=\; q_d\, s_d \;+\; q_v\, s_v \;+\; q_l\, s_l \;+\; q_i\, s_i \;, \qquad (1)$$

where specific contents in dry-air, water vapor, liquid water and ice (q_d, q_v, q_l, q_i) act as weighting factors. The common temperature T for the dry air and water vapour entropies (s_d, s_v) is possibly different from those T_l or T_i for liquid water or ice entropies (s_l, s_i), respectively.

Without lost of generality, the moist-air entropy given by (1) can be rewritten in a way similar to (B.2) in M11, leading to

$$s \;=\; q_d\, s_d \;+\; q_t\, s_v \;+\; q_l\,(s_l^* - s_v) \;+\; q_i\,(s_i^* - s_v) \;+\; q_l\,(s_l - s_l^*) \;+\; q_i\,(s_i - s_i^*) \;, \qquad (2)$$

where $q_t = q_v + q_l + q_i$ is the total water content.

The first difference from the result derived in M11 is due to s_l and s_i which must be computed in the second line of (2) at temperatures T_l and T_i, respectively, whereas s_l^* and s_i^* are computed at the common temperature T for the two gaseous species. The second line of (2) can thus be computed with $s_l - s_l^* = c_l \ln(T_l/T)$ and $s_i - s_i^* = c_i \ln(T_i/T)$, where the reference entropies $(s_l)_r$ and $(s_i)_r$ have no impact.

The other difference concerns the bracketed terms in (B.7) in M11, namely the term $R_v[q_l \ln(H_l) + q_l \ln(H_i)]$, where $H_l = e/e_{sl}$ and $H_i = e/e_{si}$ are the relative humidities with respect to liquid water and ice, respectively. These bracketed terms no longer cancel out if liquid water and ice are allowed to coexist, and/or with possible under- or super-saturations.

These differences with respect to non-mixed phase results of M11 lead to the following mixed-phase generalisation of θ_s :

$$\theta_s = \left[\theta \, \exp\!\left(- \frac{L_v\, q_l + L_s\, q_i}{c_{pd}\, T} \right) \right] \exp(\Lambda_r\, q_t) \left(\frac{T}{T_r}\right)^{\lambda\, q_t} \left(\frac{p}{p_r}\right)^{-\kappa\,\delta\, q_t} \left(\frac{r_r}{r_v}\right)^{\gamma\, q_t} \frac{(1+\eta\, r_v)^{\kappa\,(1+\delta\, q_t)}}{(1+\eta\, r_r)^{\kappa\,\delta\, q_t}} \; (H_l)^{\gamma\, q_l} \, (H_i)^{\gamma\, q_i} \left(\frac{T_l}{T}\right)^{c_l\, q_l/c_{pd}} \left(\frac{T_i}{T}\right)^{c_i\, q_i/c_{pd}} . \qquad (3)$$

The bracketed terms in the first line of (3) is the ice-liquid version of the Betts' potential temperature θ_l, where the latent heats L_v and L_s depends on T. The whole first line of (3), including the term $\exp(\Lambda_r\, q_t)$ which depends on the Third-Law reference values $(s_v)_r$ and $(s_d)_r$, forms the first-order approximation $(\theta_s)_1$. Some of the terms in the second line of (3) are used in M15 to derive the second-order approximations $(\theta_s)_2$.

The third line of (3) is made of the four new *mixed-phase correction terms*. These terms are clearly equal to unity for the non-mixed phase conditions retained in M11, namely if $T_l = T_i = T$, $H_l = 1$ for $q_l \neq 0$ and $H_i = 1$ for $q_i \neq 0$.

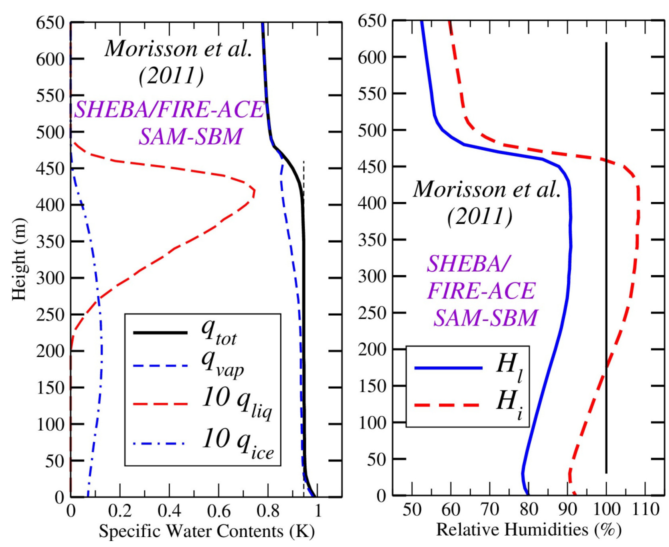

Figure 1: *The vertical profile of water species contents and relative humidities corresponding to the yellow curve (SAM-SBM) in Figure 7 of Morisson et al. (2011).*

3 Some Numerical results

The impact of the two new mixed-phase terms $(H_l)^{\gamma\, q_l}$ and $(H_i)^{\gamma\, q_i}$ in (3) are evaluated by using SHEBA/FIRE-ACE vertical profiles for (θ_l, q_t, q_l, q_i) depicted in Figure 7 of Morisson *et al.* (2011).

The profiles of (q_t, q_v, q_l, q_i) and (H_l, H_i) are shown in Fig.1. The contents in liquid water and ice are small (mind the factor 10!), but they are associated with relative humidities mostly different from 100 %. One may thus expect the factors $(H_l)^{\gamma q_l}$ and $(H_i)^{\gamma q_i}$ to be (slightly) different from unity. The vertical profiles

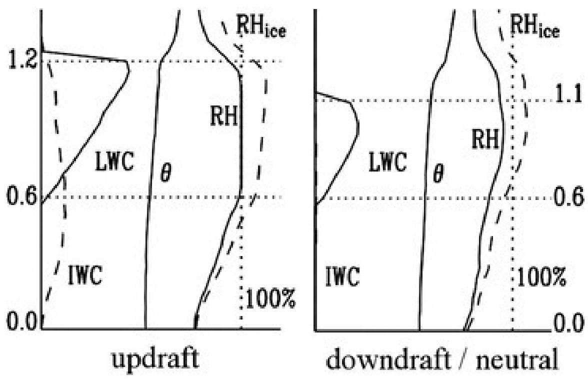

Figure 2: *The conceptual model depicted in Shupe et al. (2008) showing typical values for water species contents, θ and relative humidities in autumn Arctic mixed-phase stratiform clouds (for updraft and downdraft regions).*

$H_l(z)$ and $H_i(z)$ shown in Fig.1 are similar to those described in Fig.2 for Arctic mixed-phase clouds, with liquid and ice water content typical of updrafts and relative humidities typical of downdrafts.

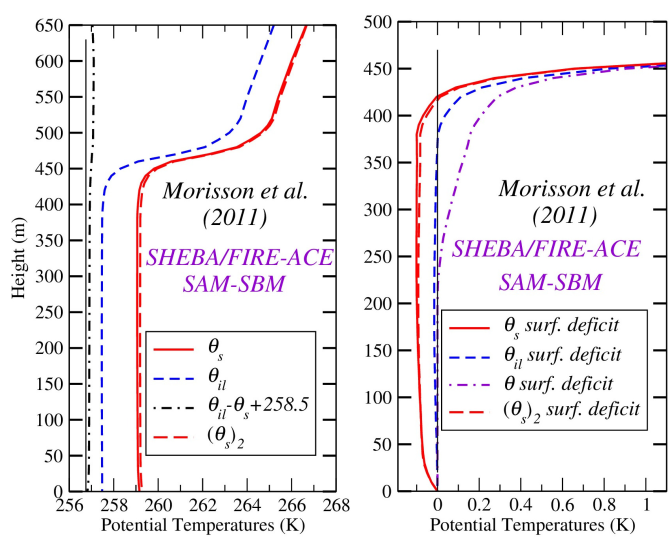

Figure 3: *Same as in Fig. 1 but for vertical profiles (left) and surface deficit values (right) for several potential temperatures: θ_s given by (3) ; θ_{il} defined in Tripoli and Cotton (1981) ; $\theta = T\,(p_0/p)^{(R_d/c_{pd})}$; the second-order value $(\theta_s)_2$ defined in M15 but multiplied by the third line of (3).*

The paradigm for describing and simulating mixed-phase cloud is to consider that the ice-liquid potential temperature θ_{il} is a conservative variables, where θ_{il} defined in Tripoli and Cotton (1981) is similar to the bracketed terms in the first line of (3), except that the latent heats $L_v(T_0)$ and $L_s(T_0)$ are computed at the triple-point temperature $T_0 = 273.16$ K (not at T).

The conserved (namely constant) feature observed for θ_{il} in the PBL of Fig.3 is likely due to the choice of the ice-liquid water static energy h_L as a prognostic variables in the SAM-SBM runs, where $h_L = c_{pd}\,T + g\,z - L_v(T_0)\,q_l - L_s(T_0)\,q_s$ is clearly a proxy for θ_{il}.

Differently, it is shown in Fig.3 that the mixed-phase moist-air entropy value θ_s given by (3) is not conserved (with $(\theta_s)_2$ being indeed a good approximations of θ_s). This may be interpreted as an impact of the term $\exp(\Lambda_r\,q_t)$ in the first line of (3) and due to changes in q_t shown in Fig.1 close to the ground (below 50 m).

This impact of q_t was missing in the definition of θ_{il} and in the approximate integration of the first and the second laws of thermodynamics derived in Dutton (1976, see before Eq.30, p.284, in the 1986 edition).

The "equivalent" version θ_{eil} defined in Tripoli and Cotton (1981) includes a factor $\exp[(L_v(T_0)q_t)/(c_{pd}T)]$ which depends on q_t, where $L_v(T_0)/(c_{pd}\,T) \approx 9$. This factor is however different from the one $\exp(\Lambda_r\,q_t)$ appearing in θ_s given by (3), where $\Lambda_r \approx 6$ depends on the Third-Law reference values $(s_v)_r$ and $(s_d)_r$. Only θ_s with $\Lambda_r \approx 6$ is an *equivalent* of the moist-air entropy.

4 Conclusions

The search for "conserved" variables based on approximations of the moist-air entropy (function or equation) should be replaced by the use of the *conservative* variables θ_s given by (3) which is a true *equivalent* variable.

A model using the mixed-phase version (3) for θ_s as a prognostic variable, including for turbulent and mass-flux mixing processes, could lead to more accurate results. The impacts of the last two terms of (3) are to be investigated (ex. for supercooled water).

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