Stochastic ensemble Kalman filter with ensemble generation using a square root form: ensemble π-algorithm

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1. Introduction

The Kalman filter algorithm is one of the most popular approaches to observation data assimilation. To implement the ensemble algorithm, the number of ensemble members must not be too large. Also, the ensemble must have a covariance matrix consistent with the covariances of analysis errors. There exist two approaches to the ensemble Kalman filter: a "stochastic filter" and a "deterministic filter". A large body of research has been made to compare stochastic and deterministic filters. As shown in several studies, the stochastic ensemble Kalman filter has advantages over a deterministic filter.

In [6], a stochastic version of the ensemble Kalman filter, which is implemented in the square root form (the ensemble π-algorithm) is proposed. The ensemble π–algorithm is what is called a stochastic filter in which an ensemble of analysis errors is generated by transforming an ensemble of forecast errors using a square root form. The transformation matrix does not depend on grid nodes. Therefore, the algorithm can be used locally, in a similar way as the LETKF algorithm [4] and can be implemented involving operations with matrices of size not greater than that of the ensemble.

A new numerical implementation scheme of the ensemble π-algorithm is proposed. In particular, we consider a more general approach than that proposed in [6] to calculate the square root of a non-symmetric matrix. Our approach is compared with the classical stochastic Kalman filter [2, 3] on the test example. The comparison showed the efficiency of proposed algorithm.

2. Ensemble π -algorithm

The ensemble π-algorithm is a stochastic filter. In this algorithm the analysis step is carried out only for the ensemble mean. The ensemble of analysis errors $\mathbf{D}$ is a matrix of dimension ($L \times N$) which columns are vectors $\{\mathbf{dx}_k^n, n=1,\ldots,N\}$, where $\mathbf{dx}_k^n \triangleq \mathbf{x}_k^{a,n} - \overline{\mathbf{x}_k^{a,n}}$, k is the number of time step, n is the ensemble number. It is obtained by transformation of the ensemble of forecast errors $\mathbf{F}$ - matrix with columns $\{\mathbf{f}_k^n, n=1,\ldots,N\} : \mathbf{f}_k^n = \mathbf{x}_k^{f,n} - \overline{\mathbf{x}_k^{f,n}}$: $\mathbf{D}^{\mathrm{T}} = (\mathbf{I} + \mathbf{\Pi}^{\mathrm{T}})^{-1}\mathbf{F}^{\mathrm{T}}$, where

$$\mathbf{\Pi}^{\mathrm{T}} = (\mathbf{C} + 0{,}25\mathbf{I})^{\frac{1}{2}} - 0{,}5\mathbf{I}, \tag{1}$$

$$\mathbf{C} = \frac{1}{N-1}\mathbf{F}^{\mathrm{T}}\mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1}(\mathbf{HF} + \mathbf{E}) = \mathbf{C}_1 + \mathbf{C}_2. \tag{2}$$

$\mathbf{E}$ is a matrix whose columns are the vectors $\boldsymbol{\varepsilon}_k^n$ - the ensemble of observational errors. More detailed description is presented in [6].

Elements of matrix $\mathbf{\Pi}$ are calculated for the ensemble $\{\mathbf{dx}_k^n, n=1,\ldots,N\}$, the matrix $\mathbf{H}$ and matrix $\mathbf{R}$ do not depend on the grid node. This makes it possible to implement the algorithm locally. At the same time the implementation of the ensemble π-algorithm requires operations with matrices whose dimension is equal to the ensemble dimension. Note that the above properties are typical of the popular LETKF algorithm [4], which is a deterministic filter.

3. Practical implementation of the ensemble π-algorithm

For implementation of the algorithm the solution of equation (1) is required, i.e. the calculation of the square root of the matrix $(\mathbf{C}+0,25\mathbf{I})$. In [6] an approximate estimate of the square root of the matrix $(\mathbf{C}+0,25\mathbf{I})$ is proposed. Let us consider a more general approach. As can be seen from formula (2), this matrix is non-symmetric. To calculate the square root of the matrix the algorithm proposed in [1] can be applied. This algorithm is based on the Schur triangular decomposition. It can be shown that the real part of the eigenvalues of the matrix $(\mathbf{C}+0,25\mathbf{I})$ is positive and the algorithm proposed in [1] can be applied to calculate the square root of this matrix.

For calculation of the inverse matrix $(\mathbf{I}+\mathbf{\Pi}^{\mathrm{T}})^{-1}$ the symmetrization using the multiplying by the transposed matrix was performed: $(\mathbf{I}+\mathbf{\Pi})(\mathbf{I}+\mathbf{\Pi}^{\mathrm{T}})\mathbf{D}^{T}=(\mathbf{I}+\mathbf{\Pi})\mathbf{F}^{T}$. To solve the resulting equation the eigenvalues and eigenvectors of a symmetric matrix were used.

There is no problem of zero eigenvalues arising during calculation of the inverse matrix in the π-algorithm. Thus, the singular value decomposition is not required as it is in the algorithm for stochastic ensemble Kalman filter proposed in [2, 3].

4. Conclusions

This article describes an efficient algorithm for data assimilation based on a stochastic ensemble Kalman filter - ensemble π-algorithm. A numerical method for the implementation of this algorithm is proposed. Numerical experiments conducted in three-dimensional domain with model data have shown the effectiveness of the ensemble π-algorithm, as compared with the algorithm of the stochastic Kalman filter proposed in [2, 3]. Root-mean-square error of analysis step of the ensemble π-algorithm in the model experiments is very close to the errors in the classical stochastic Kalman filter, and the gain in computer time becomes especially great while the number of observations used in the analyses increases.

References

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